doi:10.1006/jabr.2000.8432, available online at http://www.idealibrary.com on IDEAL®

On the Variety Determined by Symmetric Quadratic Algebras

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Communicated by J. Tits

Received October 5, 1998

We consider some polynomial identities of degree ≤ 5 which are satisfied by all symmetric quadratic algebras. We call rings satisfying these identities *generalized quadratic rings*, or *GQ-rings*. We show that when the ring is not flexible, these identities are enough to make the ring quadratic over its center. Therefore, simple nonflexible GQ-rings are symmetric quadratic algebras over their center, which is a field. For prime GQ-rings, the center has no nonzero zero divisors. Prime GQ-rings, which are not flexible, are subrings of the quadratic algebra formed by extending the center to its field of quotients. Flexible GQ-rings are noncommutative Jordan

[1]This paper was written while the second-named author held a grant from CNPq of Brazil. Part of the research was done when this author was visiting Iowa State University on a grant from FAPESP.

[2]The third-named author was partially supported by CAPES.

0021-8693/00 $35.00

rings which satisfy a quadratic condition over their commutative center. We show that any semi-prime GQ-ring is a subdirect sum of a noncommutative Jordan ring (which satisfies a quadratic condition over its commutative center) and a nonflexible ring (which is symmetric quadratic over its center).

1. INTRODUCTION

A nonassociative algebra A is said to be *quadratic over a field* K if A is an algebra over K, A has an unit element 1, and for every x in A, there are elements $t(x)$ and $n(x)$ in K such that $x^2 - t(x)x + n(x) = 0$. We remark that $t : A \to K$ is a linear form and $n : A \to K$ is a quadratic form (see Zhevlakov *et al.* [2, p. 37]). We shall say that a nonassociative ring A satisfies a *quadratic condition over a subring* U if there are functions $u,\ v,\ w : A \to U$, not all of them identically zero, such that $u(x)x^2 - v(x)x + w(x) = 0$ for each x in A. If $v : A \to U$ is linear, we shall say that A is a *quadratic ring over* U. In this paper U will either be the *center* of A defined by

$$Z(A) = \{u \in A \mid (u, A, A) = (A, u, A) = (A, A, u) = [A, u] = 0\},$$

or U will be the *commutative center* of A defined by

$$K(A) = \{u \in A \mid [A, u] = 0\}.$$

The *associator* $(a, b, c) = (ab)c - a(bc)$ and the *commutator* $[a, b] = ab - ba$ are defined as usual. The main part of the paper is to establish quadratic conditions, and then analyze the leading coefficient $u(x)$ for those cases where it is a nonzero element of some subring.

The algebra A is said to be a *symmetric quadratic algebra over* K if A is a quadratic algebra over K and $t(xy) = t(yx)$, i.e., $t([x, y]) = 0$ for all x and y in A. When A is a quadratic ring over U and $v([x, y]) = 0$ for all x and y in A, we say that A is a *symmetric quadratic ring over* U.

For a fixed field K one defines the variety generated by all symmetric quadratic algebras over K. The identities of degree 5 or less of this variety are known except for fields of characteristic 2 and 3 (see Hentzel and Peresi [7]). The identities are written with integral coefficients, and the identities are the same for all characteristics (except perhaps 2 and 3). These identities of degree 5 or less hold for all algebras satisfying an identity of type $x^2 - t(x)x + n(x) = 0$, even those where $t(x)$ and $n(x)$ are only assumed to be contained in the center, which is not necessarily a field.

A *flexitor* is an associator of the form (a, b, a) where the left and right arguments are the same. A ring A is called *flexible* if $(x, y, x) = 0$ for all $x, y \in A$. A flexible ring which satisfies the identity $(x^2, y, x) = 0$ is called *noncommutative Jordan*.

The purpose of this paper is to study the *generalized quadratic rings (GQ-rings)*, i.e., the rings satisfying some of the polynomial identities of degree ≤ 5 satisfied by all symmetric quadratic algebras.

If A is a simple GQ-ring, we prove that either A is flexible and hence noncommutative Jordan, or else A is in the variety generated by all symmetric quadratic algebras over the center of A. The center of a simple ring is a field.

If A is a prime GQ-ring, we show that either A is flexible and hence noncommutative Jordan, or else A is in the variety generated by all symmetric quadratic algebras over the quotient field of the center of A.

If A is a semi-prime GQ-ring, we establish that A is a subdirect sum with two summands: the first is a noncommutative Jordan ring (which satisfies a quadratic condition over its commutative center); the second is a nonflexible symmetric quadratic ring over its center.

2. GENERALIZED QUADRATIC RINGS

We denote by $a \circ b$ the *Jordan product* $ab + ba$, by $< a, b, c >$ the *Jordan associator* $(a \circ b) \circ c - a \circ (b \circ c)$, and by R_a the linear operator defined by $R_a(b) = a \circ b$. Let S_n be the symmetric group. We define the linear operator $\bar{S}_3$ by

$$\bar{S}_3 = \sum_{\sigma \in S_3} (-1)^\sigma R_{a_{\sigma(1)}} R_{a_{\sigma(2)}} R_{a_{\sigma(3)}},$$

where $(-1)^\sigma$ is the sign of the permutation σ.

As proved by Hentzel and Peresi [7, Theorems 1, 4, and 7], any symmetric quadratic algebra over a field of characteristic $\neq 2, 3$ satisfies the following identities, and moreover any degree ≤ 5 polynomial identity which is satisfied by all symmetric quadratic algebras over a field of characteristic 0 or characteristic > 5 is a consequence of these identities,

$$(a, a, a) = 0, \tag{1}$$

$$(a^2, a, b) - (a, a^2, b) = 0, \tag{2}$$

$$(b, a^2, a) - (b, a, a^2) = 0, \tag{3}$$

$$[(a, b, a), c] = 0, \tag{4}$$

$$((a, b, a), c, d) = 0, \tag{5}$$

$$(c, (a, b, a), d) = 0, \tag{6}$$

$$(c, d, (a, b, a)) = 0, \tag{7}$$

$$[\,[a, b] \circ [c, d], e\,] = 0, \tag{8}$$

$$([a, b] \circ c, c, d) - ([a, b], c^2, d) = 0, \tag{9}$$

$$([a, b] \circ c, d, c) - ([a, b], d, c^2) = 0, \tag{10}$$

$$\begin{aligned}[2a\,(a, b, b) - 2\,(a, b, ab) + 2b\,(b, a, a) - 2\,(b, a, ba)& \\ +a \circ (b, a, b) + b \circ (a, b, a),\ c] = 0,&\end{aligned} \tag{11}$$

$$[2a^2 \circ b^2 - (a \circ b)^2 + 2a < a, b, b > +2b < b, a, a >,\ c] = 0, \tag{12}$$

$$\bar{S}_3(a^2) - \bar{S}_3(a) \circ a = 0, \tag{13}$$

$$\begin{aligned}-(a^2 \circ c, d, e) + (a^2 \circ e, d, c) - (e, d, a^2 \circ c) + (e, d, (a \circ c) \circ a)& \\ -(c, d, (a \circ e) \circ a) + (a \circ c, d, a \circ e) - (a \circ e, d, a \circ c)& \\ -((a \circ c) \circ e, d, a) + (c, d, a^2 \circ e) + ((a \circ e) \circ c, d, a) = 0,&\end{aligned} \tag{14}$$

$$\sum_{b,c,e} \{(a, < c, d, e >, b) + (a, b, < c, d, e >)\} = 0, \tag{15}$$

where $\sum_{b,c,e}$ denotes the alternating sum on the variables b, c, e.

We shall refer to rings which satisfy identities (1)–(12) as *generalized quadratic rings*, or *GQ-rings*. If A is a GQ-ring, then it satisfies identities (4)–(7), and we have that all flexitors (x, y, x) with $x,\ y \in A$ are in $Z(A)$. We notice that any flexible GQ-ring is noncommutative Jordan. Actually, a flexible ring satisfies $(x^2, y, x) = 0$ if and only if it satisfies (3). To see this, we consider the Teichmüller identity

$$(x^2, y, x) - (x, xy, x) + (x, x, yx) - x(x, y, x) - (x, x, y)x = 0$$

which holds in any ring. Thus in a flexible ring

$$\begin{aligned}(x^2, y, x) &= -(x, x, yx) + (x, x, y)x = (yx, x, x) - (y, x, x)x \\ &= (y, x^2, x) - (y, x, x^2).\end{aligned}$$

The result follows.

EXAMPLE. Let K be any field. Consider the algebra $\mathscr{A}$ over K with basis $\{1, e_1, \ldots, e_7\}$ and the following nonzero products:

$$1e_i = e_i 1 = e_i \qquad (i = 1, \ldots, 7),$$

$$e_1e_1 = e_3, \quad e_1e_2 = e_4, \quad e_1e_4 = e_6, \quad e_2e_1 = e_5, \quad e_2e_3 = -e_6,$$

$$e_3e_2 = e_6, \quad e_4e_3 = -e_7, \quad e_5e_1 = -e_6, \quad e_5e_3 = e_7, \quad e_6e_1 = -e_7.$$

Straightforward calculations show that $\mathscr{A}$ satisfies identities (1)–(4) and, since any product with 5 elements is zero, also identities (5)–(12). Thus $\mathscr{A}$ is a GQ-ring. It is not a quadratic algebra over K since $1,\ e_1, e_1^2$ are linearly independent. It is not flexible (so it is not noncommutative Jordan) since $(e_1, e_4, e_1) = -e_7$. The center of $\mathscr{A}$ is $Z(\mathscr{A}) = \{\alpha 1 + \beta e_7 \mid \alpha,\ \beta \in K\}$. The algebra $\mathscr{A}$ is a quadratic ring over $Z(\mathscr{A})$ since if $x = \alpha 1 + \alpha_1 e_1 + \ldots + \alpha_7 e_7$ and $c(x) = \alpha$ then $e_7 x^2 - e_7 c(x) x = 0$ for all $x \in \mathscr{A}$.

The following identity shows that the defining identities of GQ-rings introduce quadratic dependence over the center in a ring which is not flexible.

THEOREM 1. *If A is a GQ-ring of characteristic $\neq 2$, then A satisfies the following quadratic condition over its center*:

$$\begin{aligned}
(x, y, x)a^2 &- \{(a, x^2, y) + (y, x^2, a) - (a \circ x, x, y) - (y, x, a \circ x)\}a \\
&- (a, (x, x, a), y) - (y, (x, x, a), a) + ((a, a, x), x, y) \\
&\qquad\qquad\qquad\qquad + (y, x, (a, a, x)) \\
&+ (a, x(ax), y) + (y, x(ax), a) - (a(xa), x, y) - (y, x, a(xa)) \\
&+ (xy, x, a^2) + (a^2, x, xy) - (xy, a \circ x, a) - (a, a \circ x, xy) \\
&- (a, x^2 y, a) = 0. \qquad (16)
\end{aligned}$$

Proof. The coefficients of the identity (16) are sums of flexitors and thus are in the center of A.

We now prove (16). Let $p(x, y, z) = 0$ denote the linearized form of (1); $p_1 = \frac{1}{2} p(x, x, a) = 0$; $p_2 = \frac{1}{2} p(a, a, x) = 0$; and $p_3 = \frac{1}{2} p(x, x, y) = 0$. Then the following identities are consequences of (1):

$$8\ y(ap_1) + 6\ a(yp_1) + 2\ a(p_1 y) - 4\ (p_1 y)a + 4\ (p_1 a)y = 0,$$

$$4\ \{(yx)p_2 + \ (xp_2)y + \ y(p_2 x) + \ p_2(xy)\} = 0,$$

$$2\ (ap_3)a + 4\ a(p_3 a) - 2\ (p_3 a)a = 0,$$

$$4\ \{[a, p(a, x, y)x] + \ [p(xa, x, y), a]\} = 0,$$

$$-2\ p((y, x, x), a, a) + 2\ [p(x^2, a, y), a] = 0,$$

$$6\ [p(ax, x, y), a] + 2\ [p(yx, a, x), a] = 0,$$

$$-4\ y\ p(ax, a, x) - \ 2\ p(x^2y, a, a)\} = 0,$$

$$4\ \{-\ p(a^2, xy, x) - \ p(xa, a, x)\ y + \ p(xy, x, a^2)\} = 0.$$

As consequences of (2) we obtain the identities

$$4\ [(a \circ x, y, x) - (y, a \circ x, x) + (a \circ y, x, x)$$

$$-(x, a \circ y, x) + (x \circ y, a, x) - (a, x \circ y, x), a] = 0,$$

$$4y\ \{-(a^2, x, x) + (x, a^2, x) - (a \circ x, a, x) + (a, a \circ x, x)\} = 0,$$

$$4\ \{-(a \circ x, x, a) + (x, a \circ x, a) - (x^2, a, a) + (a, x^2, a)\}\ y = 0.$$

Identity (3) gives the identities

$$4y\ \{(a, a \circ x, x) - (a, x, a \circ x) + (a, x^2, a) - (a, a, x^2)\} = 0,$$

$$4\ \{(x, a^2, x) - (x, x, a^2) + (x, a \circ x, a) - (x, a, a \circ x)\}\ y = 0.$$

Using (4) we obtain the identities

$$[-4\ \{(a \circ x, y, x) + (x, y, a \circ x)\} - 2\ \{(a, x \circ y, x) + (x, x \circ y, a)\}$$
$$-6\ \{(x, a \circ x, y) + (y, a \circ x, x)\}, a] = 0,$$

$$4a\ [(a, x, x) + (x, x, a), y] + 2\ [(a, x, x) + (x, x, a), y]\ a$$
$$+4y\ [(a, x, x) + (x, x, a), a] = 0,$$

$$-2a\ [(x, x, y) + (y, x, x), a] + [4\ \{(yx, x, a) + (a, x, yx)\}$$
$$-8\ \{(a, x^2, y) + (y, x^2, a)\}, a] = 0,$$

$$-4\ [[(a, x, a), x], y] + 4\ [(x, a, x), a]y - 6a\ [(x, y, x), a] = 0,$$
$$6\ [(x, a, x), y]\ a + 8[(x, y, x), a]\ a = 0.$$

Identities (5) and (7) yield

$$4\ \{((a, a, x) + (x, a, a), x, y)\} + 4\ \{((x, x, y) + (y, x, x), a, a)\}$$
$$+\ 4\ ((a, x, a), x, y) - 8\ ((x, y, x), a, a) = 0,$$
$$4\ (a, a, (x, x, y) + (y, x, x)) = 0.$$

Using identities (8), (9), and (10) we obtain

$$3\ [[a, x] \circ [x, y], a] = 0,$$

$$4\ \{([x, y] \circ a, x, a) + ([x, y] \circ x, a, a) - ([x, y], a \circ x, a)\} = 0,$$

$$-4\ \{([x, y] \circ a, x, a) - ([x, y], x, a^2)\} = 0.$$

The following identity is a partial linearization of (11):

$$\begin{aligned} -2\ [2a\ (y, x, x) + 2y\ (a, x, x) - 2\ (a, x, yx) - 2\ (y, x, ax) \\ + 2x\ (x, a, y) + 2x\ (x, y, a) - 2\ (x, a, xy) - 2\ (x, y, xa) \\ + a \circ (x, y, x) + y \circ (x, a, x) + x \circ (a, x, y) + x \circ (y, x, a), a] = 0. \end{aligned}$$

The following identity is a partial linearization of (12):

$$\begin{aligned} -[\ (a \circ x) \circ (x \circ y) - 2\ (a \circ y) \circ x^2 + 2a < x, x, y > + 2a < x, y, x > \\ + 2x < a, x, y > + 2x < a, y, x > + 2x < y, a, x > + 2y < x, a, x > \\ + 2x < y, x, a > + 2y < x, x, a >, a] = 0. \end{aligned}$$

We now add all these displayed consequences of identities (1)–(12). All the terms cancel out except the ones that gives 4 times identity (16). The result follows since the characteristic is $\neq 2$.

We now prove that the coefficient of a in the identity (16) satisfies a *symmetric condition*; i.e., we prove that it is zero when a is replaced by a commutator $[a, b]$.

LEMMA 1. *If A is a GQ-ring, then A satisfies the identity*

$$([a,b],x^2,y)+(y,x^2,[a,b])-([a,b]\circ x,x,y)-(y,x,[a,b]\circ x)=0. \tag{17}$$

Proof. We have the following identities, where $q(y, x, x) = (y, x, x) + (x, y, x) + (x, x, y)$,

$$-(ba) \circ q(y, x, x) + 2\ q(y, x, x)(ab) = 0,$$

$$p([a, b], x, y) \circ x - p(x[a, b], x, y) - p([a, b], xy, x) = 0,$$

$$q(ab, x, x) \circ y - 2\ q(ba, x, x)\ y = 0,$$

by (1);

$$-([a, b] \circ x, x, y) - (x^2, [a, b], y) + (x, [a, b] \circ x, y) + ([a, b], x^2, y) = 0,$$

$$-(x^2, y, [a, b]) - (x \circ y, x, [a, b]) + (y, x^2, [a, b]) + (x, x \circ y, [a, b]) = 0,$$

by (2);

$$(y, [a, b] \circ x, x) + (y, x^2, [a, b]) - (y, x, [a, b] \circ x) - (y, [a, b], x^2) = 0,$$

$$(x, [a, b] \circ x, y) + (x, [a, b] \circ y, x) + (x, x \circ y, [a, b])$$

$$-(x, y, [a, b] \circ x) - (x, x, [a, b] \circ y) - (x, [a, b], x \circ y) = 0,$$

by (3);

$$-[([a, b], x, y) + (y, x, [a, b]), x] + [(ba, x, x) + (x, x, ba), y] = 0,$$

$$[(x, ab, x), y] - [(x, y, x), ba] - [(x, x, y) + (y, x, x), ab] = 0,$$

by (4);

$$-([a, b] \circ x, x, y) + ([a, b], x^2, y) = 0,$$

$$-([a, b] \circ x, y, x) - ([a, b] \circ y, x, x) + ([a, b], x \circ y, x) = 0$$

by (9);

$$([a, b] \circ x, y, x) - ([a, b], y, x^2) = 0$$

by (10). We now add all these identities to obtain (17).

We now prove an identity which shows that the defining identities of flexible GQ-rings introduce quadratic dependence over the commutative center in a noncommutative ring.

THEOREM 2. *If A is a flexible GQ-ring, then A satisfies the following quadratic condition over its commutative center*:

$$\begin{aligned}
&- 6 \, (\, [a, b] \circ [a, b] \,) \, c^2 \\
&+ 6 \, (\, -[a \circ b, b] \circ [a, c] + [a \circ c, b] \circ [a, b] + 2 \, [ba, c] \circ [a, b] \,) \, c \\
&+ [ab, b] \circ [4ac + 2ca, c] + 3 \, [ab, c] \circ [ab, c] + 9 \, [ba, c] \circ [ba, c] \\
&+ 6 \, [ac, c] \circ [ba, b] - 6 \, [[a, b], c] \circ [ca, b] - 12 \, [ab, c] \circ [ba, c] \\
&+4[[a, b], bc] \circ [a, c] + [[a, c], bc - 3cb] \circ [a, b] + 12[ba, c^2] \circ [a, b] \\
&- 4 \, [c[a, b], b] \circ [a, c] + [-4c.ac + 6c^2 a - 8c.ca, b] \circ [a, b] \\
&+ [-18ba.c - 8ca.b - 6c.ab - 4ac.b, c] \circ [a, b] = 0. \qquad (18)
\end{aligned}$$

Proof. The linearized form of the flexible law is $f(x, y, z) = (x, y, z) + (z, y, x) = 0$. We list the following consequences of the flexible law:

$$-2[f(ac, b, [a, b]), c] = 0, \qquad 2[f([a, b], b, ca), c] = 0,$$

$$8f([a, b], cb, [a, c]) = 0, \qquad 4f([[a, b], c], b, ca) = 0,$$

$$f(2c.ca - 8ca.c + 2ac.c + 4c.ac, b, [a, b]) = 0,$$

$$f(-6c.ba + 6ca.b - 2b.ca + 24ba.c - 4b.ac - 18ab.c, c, [a, b]) = 0,$$

$$f(4b.ba - 6ba.b + 6b.ab - 4ab.b, c, [a, c]) = 0,$$

$$-6f(c^2, a, b) \circ [a, b] = 0, \qquad 6\{bf(a, c, c)\} \circ [a, b] = 0,$$

$$12\{-f(b, a, c) \circ [a, b] + ([a, b], c, [a, b])\}c = 0,$$

$$-4f([[a, b], c], b, ac) = 0, \qquad 6\{f(b, a, c)c\} \circ [a, b] = 0,$$

$$6[a, b]\{f(b, a, c) \circ c\} = 0, \qquad 4[a, c]f(b, [a, b], c) = 0,$$

$$-6(c, a, c)\{[ba, b] + ab.b\} = 0, \qquad 6(c, a, c) \circ (b.ba) = 0,$$

$$12\{(b, a, b)[a, c]\} \circ c = 0, \qquad 12(b, a, b)\{c.ca - ac.c\} = 0,$$

$$f(-2ab.b + 2b.ba, [a, c], c) = 0, \qquad -2f(b[a, c], [a, b], c) = 0,$$

$$-2[a, b]f([a, c], b, c) = 0, \qquad 18\{[a, b](c, ab, c) + (c, ba, c)[a, b]\} = 0,$$

$$6f(b, a \circ c, c)[a, b] = 0, \qquad 10f([a, c], (b, a, b), c) = 0,$$

$$-2f([a, c], c, b)[a, b] = 0, \qquad 4f([a, b], c, b)[a, c] = 0,$$

$$4[a, c]f([a, b], b, c) = 0, \qquad 2f([a, b], [a, c]b, c) = 0,$$

$$8[a, b]f(b, ca, c) = 0, \qquad -6(c[a, b])f(b, a, c) = 0,$$

$$6f(ab, c, c)[a, b] = 0, \qquad 6[a, b]f(ba, c, c) = 0,$$

$$4[a, b]f(b, ac, c) = 0, \qquad 18([a, b], c^2, [a, b]) = 0,$$

$$-6(b(ab))(c, a, c) = 0, \qquad 6f(b, a, c)([a, b]c) = 0,$$

$$10[(b, a, b), c] \circ [a, c] = 0, \qquad 6\{(c, a, c) \circ b\}[a, b] = 0.$$

We now list the following consequences of the Jordan law $(x^2, y, x) = 0$:

$$-12\ \{([a, b] \circ c, [a, b], c) + (c^2, [a, b], [a, b])\} = 0,$$

$$-4\ \{([a, b] \circ [a, c], b, c) + ([a, b] \circ c, b, [a, c]) + ([a, c] \circ c, b, [a, b])\} = 0.$$

Next, we list the following consequences of identity (8):

$$6[[a,b]\circ[a,b],c^2]=0, \qquad 2[[ab,b]\circ[a,c],c]=0,$$
$$3[[a,c]\circ[b,c],[a,b]]=0.$$

Finally, we list the following consequences of identity (9):

$$4\{([a,b]\circ b,c,[a,c])+([a,b]\circ c,b,[a,c])-([a,b],b\circ c,[a,c])\}=0,$$

$$12\{([a,b]\circ c,c,[a,b])-([a,b],c^2,[a,b])\}=0,$$

$$8\{([a,b]\circ[a,c],b,c)+([a,b]\circ b,[a,c],c)-([a,b],[a,c]\circ b,c)\}=0,$$

$$4\{([a,c]\circ b,c,[a,b])+([a,c]\circ c,b,[a,b])-([a,c],b\circ c,[a,b])\}=0,$$

$$-4\{([a,c]\circ[a,b],b,c)+([a,c]\circ b,[a,b],c)-([a,c],[a,b]\circ b,c)\}=0.$$

By adding together all the identities listed above and simplifying we obtain -1 times (18). From (8) we have that $[a,b]\circ[c,d]$ is in the commutative center for all $a,\ b,\ c,\ d\in A$. This means that all the coefficients of (18) are in the commutative center. This finishes the proof of Theorem 2.

Remark. Using the method described by Hentzel and Jacobs [4] and its implementation, the computer program Albert (see Jacobs *et al.* [5, 6]), we verified that any GQ-ring satisfies the shorter quadratic condition

$$(x,y,x)a^2-\{(a,x^2,y)+(y,x^2,a)-(a\circ x,x,y)-(y,x,a\circ x)\}a$$

$$+(ay,a,x^2)+(x^2,a,ay)-(ay,a\circ x,x)-(x,a\circ x,ay)-(x,a^2y,x)=0$$

over its center. We also verified that if a GQ-ring is flexible and satisfies both identities (13) and (15), then it satisfies the shorter quadratic condition

$$([a,b]\circ[a,b])c^2-2(-[ab,a]\circ[b,c]-[bc,a]\circ[a,b]+[ba,c]\circ[b,a])c$$

$$+[a^2,bc]\circ[b,c]+[bc^2,a]\circ[a,b]-[ac,b]\circ[bc,a]-[ca,b]\circ[bc,a]=0,$$

over its commutative center. We do not give proofs of these shorter quadratic conditions here because these proofs are longer than the ones presented in Theorems 1 and 2.

3. PRIME RINGS

A ring is said to be *prime* if whenever I and J are ideals of the ring and $IJ = 0$, then either $I = 0$ or $J = 0$.

The center $Z(A)$ of a prime ring A has no nonzero zero divisors. If $Z(A)$ is not zero, then it has a field of quotients $\tilde{Z}(A)$. The $\tilde{Z}(A)$-algebra $A^* = \tilde{Z}(A) \otimes_{Z(A)} A$ contains an isomorphic copy of A. Since the identities defining GQ-rings are homogeneous, A^* will be a GQ-algebra, whenever A is a GQ-ring with nonzero center.

THEOREM 3. *If A is a prime GQ-ring of characteristic $\neq 2$ which is not flexible, then A^* is a symmetric quadratic algebra over $\tilde{Z}(A)$. Furthermore, A is in the variety determined by all symmetric quadratic algebras over $\tilde{Z}(A)$.*

Proof. Since A is not flexible, then some flexitor (x, y, x) is a nonzero element of $Z(A)$. Since the coefficient (x, y, x) is invertible in $\tilde{Z}(A)$, it follows from Theorem 1 that A satisfies a quadratic identity over the field $\tilde{Z}(A)$. Thus A^* is a quadratic algebra over $\tilde{Z}(A)$. By Lemma 1, A^* is a symmetric quadratic algebra.

Since A is a subalgebra of a symmetric quadratic algebra over $\tilde{Z}(A)$, it satisfies all identities that hold for all symmetric quadratic algebras over $\tilde{Z}(A)$, and so A is in the variety determined by all symmetric quadratic algebras over $\tilde{Z}(A)$.

If the center of a simple ring is not zero then it is a field (see Zhevlakov *et al.* [2, p. 137]). Thus we have:

COROLLARY 1. *If A is a simple GQ-ring of characteristic $\neq 2$ which is not flexible, then A is in the variety determined by all symmetric quadratic algebras over the center of A.*

4. SEMI-PRIME RINGS

A ring is called *semi-prime* if the only ideal I such that $I^2 = 0$ is $I = 0$.

In this section we examine the case when the ring A is a semi-prime GQ-ring of characteristic $\neq 2$. We let F be the additive span of all flexitors (x, y, x) with x and y in A, and $I = \{a \in A \mid Fa = 0\}$. Since $F \subset Z(A)$ we have that $F + FA$ and I are ideals of A. Thus $I \cap (F + FA)$ is also an ideal of A which squares to zero. The semi-prime condition implies that $I \cap (F + FA) = 0$. Therefore A is a subdirect sum of $A/(F + FA)$ and A/I. The factor $A/(F + FA)$ is flexible and so is a noncommutative Jordan ring. It satisfies the quadratic condition over its commutative center mentioned in Theorem 2. It follows from Theorem 1 and Lemma 1 that the factor A/I

is a symmetric quadratic ring over $Z(A/I)$. We have proved the following theorem.

THEOREM 4. *Any semi-prime GQ-ring A of characteristic $\neq 2$ is a subdirect sum of two rings. The first summand is a noncommutative Jordan ring satisfying a quadratic condition over its commutative center. The second summand is a symmetric quadratic ring over its center.*

ACKNOWLEDGMENTS

We used the computer facilities at Iowa State University (Project Vincent), at IME-USP (Instituto de Matemática e Estatística), and LCCA-USP (Laboratório de Computação Científica Avançada). The second-named author used a X-Terminal granted by FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo).

REFERENCES

1. J. M. Osborn, Quadratic division algebras, *Trans. Amer. Math. Soc.* **105** (1962), 202–221.
2. K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, "Rings That Are Nearly Associative," Academic Press, New York, 1982.
3. A. Elduque, Quadratic alternative algebras, *J. Math. Phys.* **31** (1990), 1–5.
4. D. P. Jacobs, and I. R. Hentzel, A dynamic programming method for building free algebras, *Comput. Math. Appl.* **22** (1991), 61–66.
5. D. P. Jacobs, S. V. Muddana, and A. J. Offutt, A computer algebra system for nonassociative identities, *in* "Proceedings of the Fifth Internationl Conference on Hadronic Mechanics and Nonpotential Interactions" (H. C. Myung, Ed.), Nova Science Publishers, Inc., New York, 1993.
6. D. P. Jacobs, D. Lee, S. V. Muddana, A. J. Offutt, K. Prabhu, and T. Whiteley, Albert's user guide, Department of Computer Science, Clemson University, 1993.
7. I. R. Hentzel, and L. A. Peresi, Degree three, four and five identities of quadratic algebras, *J. Algebra* **206** (1998), 1–16.